

# A nonlinear formulation for the earthquake responses of dam-reservoir systems

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## ABSTRACT

A nonlinear formulation for fluid-structure interactions based on the velocity potential is developed. In this formulation the convective accelerations, nonlinear surface waves and exact transmitting boundary condition are included. Nonlinear behaviors of dam-reservoir systems subjected to constant accelerations, harmonic and actual earthquake ground motions are also investigated.

## INTRODUCTION

The linear analysis of hydrodynamic pressures acting on rigid dams was first reported by Westergaard (1933). By neglecting the water compressibility, Chwang (1983) addressed the nonlinear hydrodynamic pressures on a rigid plate when the system was subjected to a short period of constant accelerations. Hung and Wang (1987), using the finite difference method and primitive variables for the governing equations, obtained the nonlinear hydrodynamic pressure on a rigid dam subjected to ground motions.

In this paper a nonlinear formulation, based on the velocity potential, for fluid-structure interactions is proposed. In the formulation the convective terms, nonlinear surface waves and exact transmitting boundary conditions, developed by Tsai et al. (1990a, 1990b, 1990c), are included. The reservoir is divided into two fields, the near and far fields (see Fig. 1). The near field is considered as a nonlinear area, and the behavior in the far field is linear. The final discretized matrices are symmetrical, even the nonlinearity of the near field are involved. The nonlinear responses of the system subjected to constant accelerations, harmonic and actual earthquake ground motions are also presented.

## GOVERNING EQUATION FOR THE NEAR FIELD

Assuming that water is inviscid, the equations of motion, in terms of primitive variables, for the reservoir can be written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \quad (1)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g \quad (2)$$

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The Bernoulli equation can be used to replace the momentum equations, if irrotational wave is assumed; that is,

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{q^2}{2} + gz = 0 \quad (3)$$

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{1}{\rho C^2} \frac{\partial P}{\partial t} \quad (4)$$

where  $q^2 = (\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial z})^2$ ,  $u = \frac{\partial \phi}{\partial x}$ ,  $w = \frac{\partial \phi}{\partial z}$ ,  $\rho$  = the mass density,  $\phi$  = the velocity potential,  $g$  = the gravity acceleration,  $C$  = the velocity of sound in water,  $u$  and  $v$  = the velocities of the fluid in a system of coordinates  $x$  and  $z$ , and  $P$  = the pressure.

Differentiation of Eq. 3 with respect to  $t$  results in

$$-\frac{1}{\rho} \frac{\partial P}{\partial t} = \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} \left( \frac{q^2}{2} \right) \quad (5)$$

Substitution of Eq. 4 into Eq. 5 yields the nonlinear governing equation given by

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{1}{C^2} \frac{\partial}{\partial t} \left( \frac{q^2}{2} \right) \quad (6)$$

Eq. 6 can be rewritten, in terms of the potential velocity  $\phi$ , as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{1}{C^2} \frac{\partial}{\partial t} \left( \frac{q^2}{2} \right) \quad (7)$$

### BOUNDARY CONDITIONS OF THE NEAR FIELD

Two boundary conditions for the free surface are given as follows.

(i) the kinematic boundary condition

$$\frac{\partial \phi}{\partial n} = \frac{\partial \eta}{\partial t} n_z \quad (8)$$

(ii) the dynamic boundary condition

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + g\eta = 0 \quad (9)$$

The other boundary conditions are

$$\frac{\partial \phi}{\partial n} = \bar{u}_n \quad (10)$$

and

$$\phi = \bar{\phi} \quad (11)$$

where  $\eta$  is the displacement of the free surface in the  $z$  direction;  $n_z$  is the  $z$ -component of the unit normal direction,  $n$ , on the free surface.

If we set

$$\eta^* = \eta n_z \quad (12)$$

then its derivative with respect to time  $t$  is

$$\frac{\partial \eta^*}{\partial t} = \frac{\partial \eta}{\partial t} n_z + \eta \frac{\partial n_z}{\partial t} \quad (13)$$

Substitution of Eq. 13 into Eq. 8 results in the kinematic boundary condition on the free surface; that is,

$$\frac{\partial \phi}{\partial n} = \frac{\partial \eta^*}{\partial t} - \eta \frac{\partial n_z}{\partial t} \quad (14)$$

Substitution of Eq. 12 into the dynamic boundary condition, Eq. 9, yields

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \frac{g}{n_z} \eta^* = 0 \quad (15)$$

Applying the Galerkin's method to Eq. 7, one obtains

$$\int_A \mathbf{N}^T \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) dA = \frac{1}{C^2} \int_A \mathbf{N}^T \frac{\partial^2 \phi}{\partial t^2} dA + \frac{1}{C^2} \int_A \mathbf{N}^T \left[ \frac{\partial}{\partial t} \left( \frac{q^2}{2} \right) \right] dA \quad (16)$$

Integration of the term on the left hand side by parts yields

$$\begin{aligned} & \int_S \mathbf{N}^T \frac{\partial \phi}{\partial n} dS - \int_A \left( \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial z} \frac{\partial \phi}{\partial z} \right) dA \\ &= \frac{1}{C^2} \int_A \mathbf{N}^T \frac{\partial^2 \phi}{\partial t^2} dA + \frac{1}{C^2} \int_A \mathbf{N}^T \left[ \left( \frac{\partial \phi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \phi}{\partial z} \frac{\partial}{\partial z} \right) \frac{\partial \phi}{\partial t} \right] dA \end{aligned} \quad (17)$$

Introduction of the shape function  $\mathbf{N}$  into Eq. 17 leads to

$$\begin{aligned} & \int_A \left( \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial z} \frac{\partial \mathbf{N}}{\partial z} \right) \bar{\Phi} dA + \frac{1}{C^2} \int_A \mathbf{N}^T \mathbf{N} dA \bar{\Phi} \\ &= \int_S \mathbf{N}^T \frac{\partial \phi}{\partial n} dS - \frac{1}{C^2} \int_A \mathbf{N}^T \left[ \left( \frac{\partial \phi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \phi}{\partial z} \frac{\partial}{\partial z} \right) \frac{\partial \phi}{\partial t} \right] dA \end{aligned} \quad (18)$$

Rearranging Eq. 18, the following equation can be obtained

$$\mathbf{M} \bar{\Phi} + \mathbf{K} \bar{\Phi} = \mathbf{B} - \mathbf{E} \quad (19)$$

The matrices  $\mathbf{M}$ ,  $\mathbf{K}$ ,  $\mathbf{B}$  and  $\mathbf{E}$  in Eq. 19 are defined as follows

$$\mathbf{M} = \frac{1}{C^2} \int_A \mathbf{N}^T \mathbf{N} dA \quad (20)$$

$$\mathbf{K} = \int_A \left( \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial z} \frac{\partial \mathbf{N}}{\partial z} \right) dA \quad (21)$$

$$\mathbf{B} = \int_S \mathbf{N}^T \frac{\partial \phi}{\partial n} dS \quad (22)$$

and

$$\mathbf{E} = \frac{1}{C^2} \int_A \mathbf{N}^T \left[ \left( \frac{\partial \phi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \phi}{\partial z} \frac{\partial}{\partial z} \right) \frac{\partial \phi}{\partial t} \right] dA \quad (23)$$

Applying the Galerkin's method to Eq. 15, one obtains

$$\int_{S_2} \mathbf{N}^T \frac{\partial \phi}{\partial t} dS + \int_{S_2} \mathbf{N}^T \left( \frac{q^2}{2} \right) dS + g \int_{S_2} \mathbf{N}^T \frac{1}{n_z} \eta^* dS = 0 \quad (24)$$

Introduction of the shape function into Eq. 24 yields

$$\int_{S_2} \mathbf{N}^T \mathbf{N} dS \dot{\Phi} + \int_{S_2} \mathbf{N}^T \left( \frac{q^2}{2} \right) dS + g \int_{S_2} \frac{1}{n_z} \mathbf{N}^T \mathbf{N} \underline{\eta}^* dS = 0 \quad (25)$$

For convenience, Eq. 25 can be rewritten in the following matrix form

$$\mathbf{H} \dot{\Phi} + \mathbf{G} \underline{\eta}^* = -\mathbf{L} \quad (26)$$

where

$$\mathbf{H} = \int_{S_2} \mathbf{N}^T \mathbf{N} dS \quad (27)$$

$$\mathbf{G} = g \int_{S_2} \frac{1}{n_z} \mathbf{N}^T \mathbf{N} dS \quad (28)$$

$$\mathbf{L} = \frac{1}{2} \int_{S_2} \mathbf{N}^T (q^2) dS = \frac{1}{2} \int_{S_2} \mathbf{N}^T \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] dS = \frac{1}{2} \int_{S_2} \mathbf{N}^T \left[ \left( \frac{\partial \eta}{\partial t} n_z \right)^2 + \left( \frac{\partial \phi}{\partial s} \right)^2 \right] dS \quad (29)$$

Substituting Eq. 14 into Eq. 22, the boundary condition on the free surface is given by

$$\begin{aligned} \mathbf{B}_2 &= \int_{S_2} \mathbf{N}^T \frac{\partial \phi}{\partial n} dS = \int_{S_2} \mathbf{N}^T \frac{\partial \eta^*}{\partial t} dS - \int_{S_2} \mathbf{N}^T \eta \frac{\partial n_z}{\partial t} dS \\ &= \int_{S_2} \mathbf{N}^T \mathbf{N} dS \underline{\dot{\eta}}^* - \int_{S_2} \mathbf{N}^T \eta \frac{\partial n_z}{\partial t} dS = \mathbf{H} \underline{\dot{\eta}}^* - \mathbf{V} \end{aligned} \quad (30)$$

where

$$\mathbf{V} = \int_{S_2} \mathbf{N}^T \eta \frac{\partial n_z}{\partial t} dS \quad (31)$$

Combining 19 and 26, the following matrix can be obtained

$$\begin{aligned} &\begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 \\ M_{21} & M_{22} & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\Phi}_1 \\ \ddot{\Phi}_2 \\ \ddot{\Phi}_3 \\ \underline{\dot{\eta}}^* \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\mathbf{H} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\Phi}_1 \\ \dot{\Phi}_2 \\ \dot{\Phi}_3 \\ \underline{\dot{\eta}}^* \end{Bmatrix} \\ &+ \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ K_{31} & K_{32} & K_{33} & 0 \\ 0 & 0 & 0 & -\mathbf{G} \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \underline{\eta}^* \end{Bmatrix} = \begin{Bmatrix} \mathbf{B}_1 - \mathbf{E}_1 \\ \mathbf{B}_2 - \mathbf{E}_2 \\ \mathbf{B}_3 - \mathbf{E}_3 \\ \mathbf{L} \end{Bmatrix} \end{aligned} \quad (32)$$

Substitution of Eq. 30 into Eq. 32 leads to

$$\begin{aligned} &\begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 \\ M_{21} & M_{22} & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\Phi}_1 \\ \ddot{\Phi}_2 \\ \ddot{\Phi}_3 \\ \underline{\dot{\eta}}^* \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{H} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathbf{H} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\Phi}_1 \\ \dot{\Phi}_2 \\ \dot{\Phi}_3 \\ \underline{\dot{\eta}}^* \end{Bmatrix} \\ &+ \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ K_{31} & K_{32} & K_{33} & 0 \\ 0 & 0 & 0 & -\mathbf{G} \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \underline{\eta}^* \end{Bmatrix} = \begin{Bmatrix} \mathbf{B}_1 - \mathbf{E}_1 \\ -\mathbf{V} - \mathbf{E}_2 \\ \mathbf{B}_3 - \mathbf{E}_3 \\ \mathbf{L} \end{Bmatrix} \end{aligned} \quad (33)$$

It should be noted that nodes on the free surface are denoted by subscript 2, at the interface of the near and far fields by 3. The remaining nodes are denoted by subscript 1.

### EXACT TRANSMITTING BOUNDARY CONDITION

The effect of radiation damping in the time-domain analyses is treated following the development by Tsai et al. (1990a, 1990b, 1990c). The procedure will be described briefly in the following :

The governing equation is

$$\nabla^2 \phi = \frac{1}{C^2} \ddot{\phi} \quad (34)$$

The velocity potential should satisfy : (1) the compatibility equations at the interface of the near and far fields, (2) the rigid boundary condition at the floor of the reservoir, (3) the radiation condition for the infinite fluid domain, (4) the boundary condition on the free surface,  $\frac{\partial \phi}{\partial t} = 0$ , if the surface wave is neglected, and (5) zero initial conditions for the velocity potential.

The solution of the velocity potential,  $\Phi_3(t)$ , at the interface of the near and far fields, is therefore given by

$$\Phi_3(t) = -C \int_0^t \Psi \Xi(\tau) \Psi^T G_3 \frac{\partial \Phi_3(\tau)}{\partial n} d\tau \quad (35)$$

where  $\Xi(\tau)$  is an  $M \times M$  diagonal matrix with  $m$ th diagonal term  $= J_0(\lambda_m C(t - \tau))$ ,  $t > 0$ .  $\Psi$  is the normalized modal matrix with respect to  $G_3$  at the interface of the near and far fields.  $\lambda_m$  is the  $m$ th eigenvalue associated with  $m$ th eigenvalue.  $G_3 = \int_{S_3} N_3^T N_3 dS$ . The time axis is divided into  $N$  equal intervals  $\Delta t$ , then Eq. 35 can be rewritten in the following form

$$\Phi_3(N\Delta t) = F(N\Delta t) - R G_3 \frac{\partial \Phi_3(N\Delta t)}{\partial n} \quad (36)$$

For the special case of constant temporal variation and zero initial conditions, the matrices  $F$  and  $R$  are defined as

$$F(N\Delta t) = \sum_{n=1}^{N-1} \Psi Q \Psi^T G_3 \frac{\partial \Phi_3(n\Delta t)}{\partial n} \quad (37)$$

and

$$R = \Psi \Upsilon \Psi^T \quad (38)$$

where  $Q$  is an  $M \times M$  diagonal matrix with  $m$ th diagonal term  $Q_{mm}$

$$Q_{mm} = \frac{-1}{2\lambda_m} \int_{\lambda_m C(N-n-1)\Delta t}^{\lambda_m C(N-n+1)\Delta t} J_0(\tau) d\tau = \frac{-1}{2\lambda_m} \left[ \frac{\pi\tau}{2} \{J_0(\tau)H_{-1}(\tau) + H_0(\tau)J_1(\tau)\} \right]_{\tau=\lambda_m C(N-n-1)\Delta t}^{\tau=\lambda_m C(N-n+1)\Delta t} \quad (39)$$

and  $\Upsilon$  is also an  $M \times M$  diagonal matrix with  $m$ th diagonal term  $\Upsilon_{mm}$  given by

$$\Upsilon_{mm} = \frac{1}{2\lambda_m} \int_0^{\lambda_m C\Delta t} J_0(\tau) d\tau = \frac{1}{2\lambda_m} \left[ \frac{\pi\tau}{2} \{J_0(\tau)H_{-1}(\tau) + H_0(\tau)J_1(\tau)\} \right]_{\tau=0}^{\tau=\lambda_m C\Delta t} \quad (40)$$

where  $H_\nu(\tau)$  is the Struve's function of order  $\nu$ .

It is noted that coefficients in the matrix  $R$  are constant if the time step  $\Delta t$  is constant. Substitution of Eqs. 22 and 36 into Eq. 33 yields

$$\begin{aligned}
& \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 \\ M_{21} & M_{22} & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\Phi}_1 \\ \ddot{\Phi}_2 \\ \ddot{\Phi}_3 \\ \ddot{\eta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -H \\ 0 & 0 & 0 & 0 \\ 0 & -H & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \eta \end{Bmatrix} \\
& + \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ K_{31} & K_{32} & (K_{33} + R^{-1}) & 0 \\ 0 & 0 & 0 & -G \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \eta \end{Bmatrix} = \begin{Bmatrix} B_1 - E_1 \\ -V - E_2 \\ R^{-1}F - E_3 \\ L \end{Bmatrix} \quad (41)
\end{aligned}$$

It should be noted that the matrices of the system in Eq. 41 are symmetrical although the convective terms, nonlinear surface waves and exact transmitting boundary condition are included. The incremental form of the formulation can be obtained from Eq. 41. Iteration procedures are adopted for each time step in this study to obtain numerical solutions.

## NUMERICAL EXAMPLES

Hydrodynamic pressures, in excess of the hydrostatic pressure, acting on a vertical rigid dam, having a reservoir of 180m in height with flat floor extending to infinity, subjected three different types of loadings are given to illustrate the nonlinear behaviors of the dam-reservoir system. The extending length of the near field in the upstream direction and the sound velocity in water are taken as 360m and 1438.656 m/sec respectively.

### 1. Constant Accelerations

When the system is excited by a constant acceleration,  $a = 0.5g$ , very significant nonlinear effects, shown in Fig. 2, are observed. The hydrodynamic pressure for the nonlinear case is much higher than that for the linear case. This is because the nonlinear effect, shown in Eq. 3, is proportional to the square of the velocity which is monotonously increasing with time for this special case of constant accelerations.

### 2. Harmonic Ground Motions

If the system is subjected to a harmonic ground motion,  $a \sin 12t = 0.5g \sin 12t$ , the results in Fig. 3 show insignificant nonlinear effects when compared to linear responses. In this case the velocities of the fluid particles along the dam face are also a harmonic function. Therefore, the nonlinear effect is insignificant, even though the excitation frequency, 12rad/sec, is near the first natural frequency of the reservoir, 12.555 rad/sec.

### 3. Earthquake Ground Motions

An actual earthquake ground motion, 1940 El Centro, is applied to the dam-reservoir system. The response of the system, shown in Fig. 4, also shows negligible nonlinear effect.

## CONCLUSIONS

A new nonlinear formulation for the fluid-structure interactions is presented. The proposed formulations based on the velocity potential yield symmetrical matrices, although convective accelerations, nonlinear surface waves and the exact transmitting boundary condition are involved. The nonlinear behaviors of the dam-reservoir system subjected to different type of ground motions in the upstream-downstream direction are also examined. The nonlinear effects for the two dimensional case of the rigid dam-reservoir system are very much dependent on the loading types.

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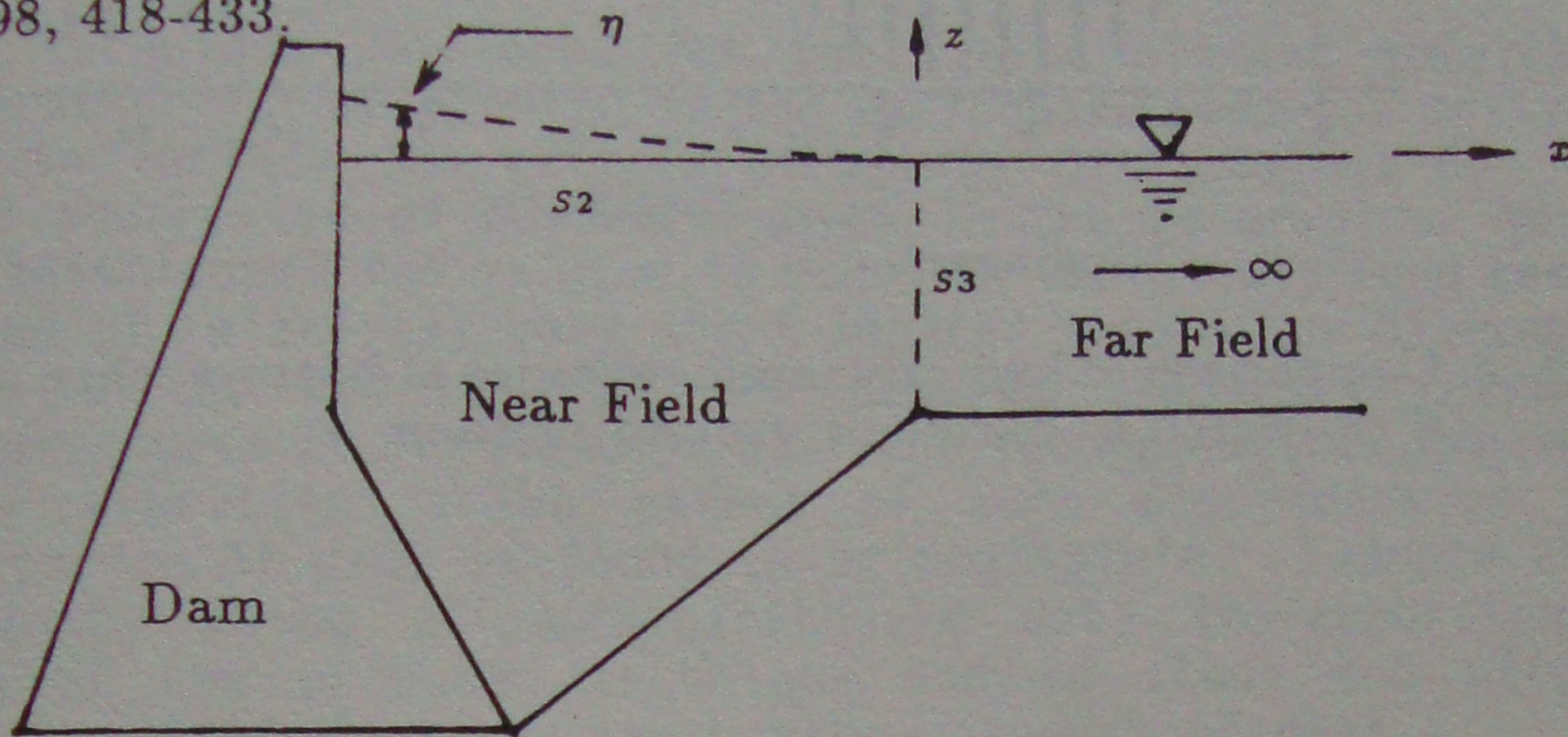


Fig. 1 Dam-Reservoir System

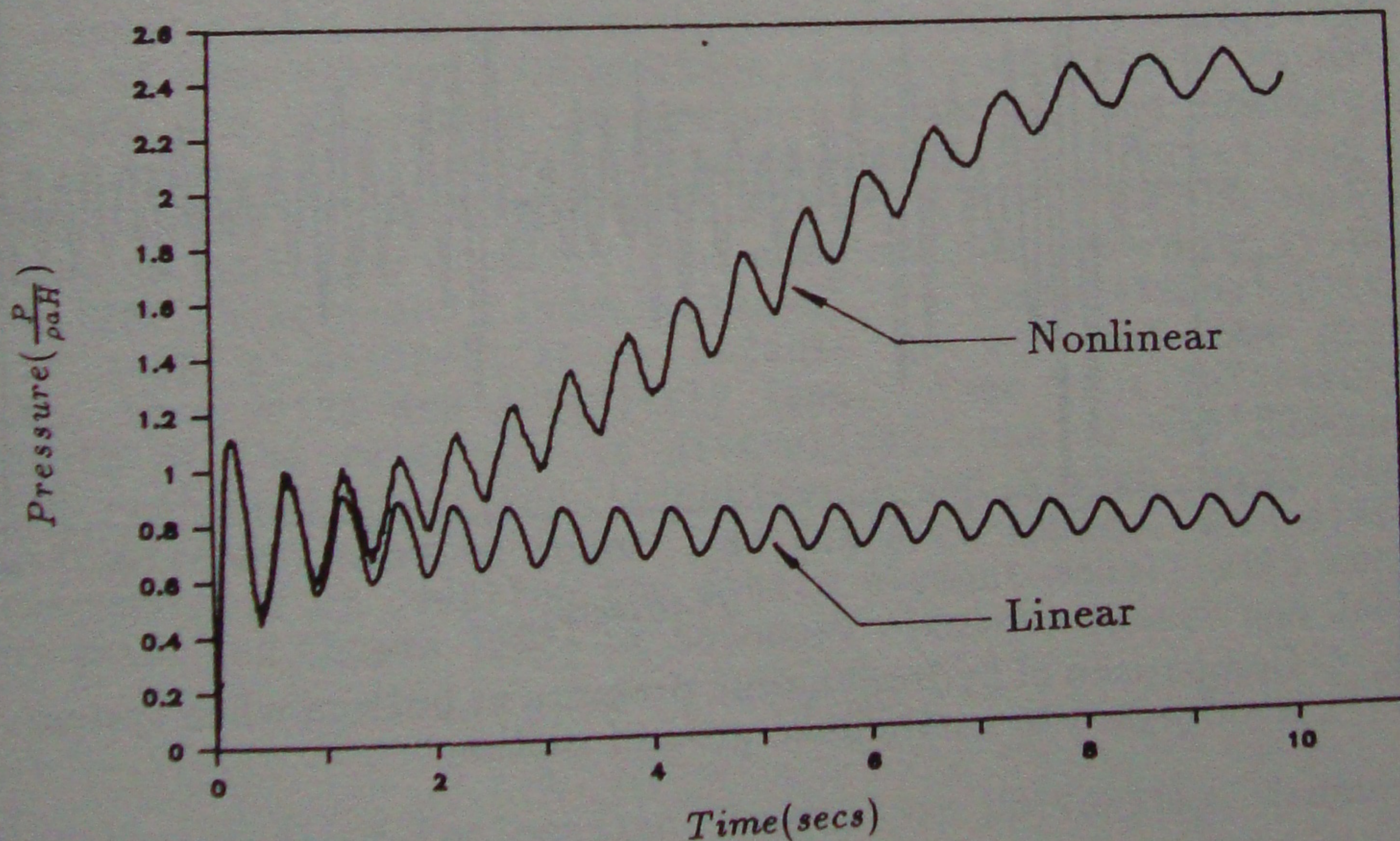


Fig. 2 Comparison of hydrodynamic pressure at bottom when system is subjected to constant acceleration

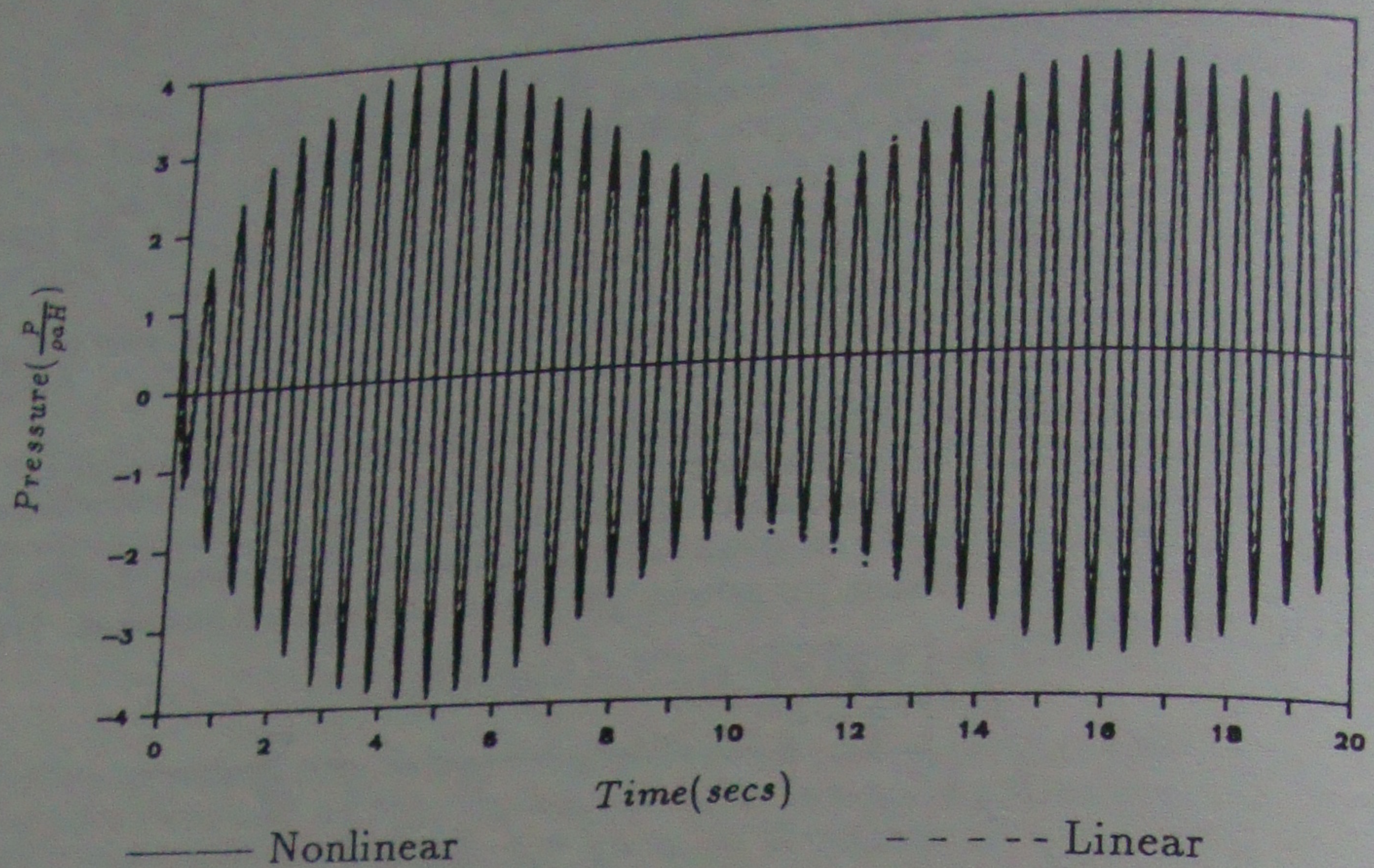


Fig. 3 Comparison of hydrodynamic pressure at bottom when system is subjected to harmonic ground motion

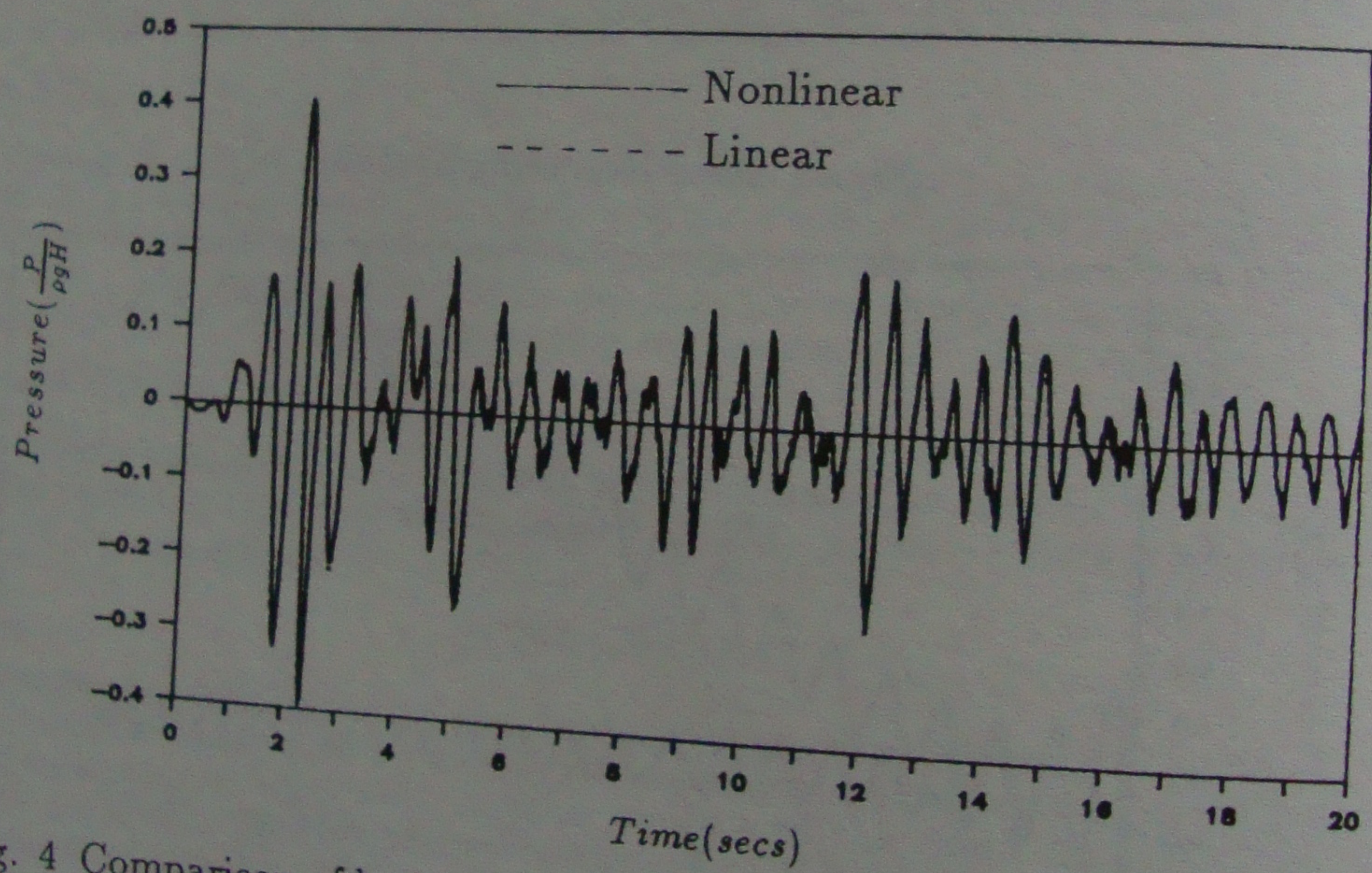


Fig. 4 Comparison of hydrodynamic pressure at bottom when system is subjected to earthquake ground motion